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A New Method for Computing Continuous Functions with Fuzzy Variable

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Abstract: This study proposes a new method for computing f(U) where f is a real continuous function and U is a fuzzy interval. The computation of f(U) is performed by incorporating optimisation technique into Zadeh's extension principle. By discretising α up to n finite numbers, a set of n closed and bounded intervals is obtained. Here, the computation of f on closed and bounded intervals is the same idea of solving unconstrained optimisation problems. For every finite numbers of α, if the function to be optimised is unimodal, the authors apply Brent's method. One of the main advantages of using this method is that it does not require the calculation of derivative. In case where f is reduced to monotone or to a straight line, the optimal solutions are obtained at the endpoints of intervals. This new strategy gives better results and requires only few function evaluations. An example is provided to illustrate the effectiveness of the proposed method.

Key words: Fuzzy set, fuzzy interval, discretisation, optimisation, zadeh's extension principle

INTRODUCTION

Fuzzy set theory has witnessed an exponential development since its introduction in 1965 (Zadeh, 1965), both from the theoretical and applied point of view. One of the most fundamental principles of fuzzy set theory is Zadeh's extension principle. It provides a general way of extending a real function of one or more variables to a function accepting fuzzy sets on the real line as arguments. In general, the practical use of Zadeh's extension principle can be quite complicated. Of particular interest is the case of continuous functions and fuzzy intervals as arguments. In that case, the extension principle can be performed in a parallel manner, by computing the α-cuts of the output, which turns out to be a fuzzy interval as well. This observation allows researcher to carry over interval arithmetic to fuzzy interval arithmetic. If the function is monotone, then the endpoints of the output can be determined quite easily, as is for instance the case for the addition of fuzzy intervals. However, the difficulty arises when the function is non-monotone. In that case, the computation of the α-cuts is not an easy task (Chalco-Cano *et al*., 2009).

In the literature, research on computational methods proposes several approaches to fulfil the requirements of Zadeh's extension principle. For instance, Kaufmann and Gupta (1991) have described an analytical method based on α-cuts and interval arithmetic. Unfortunately, the results are not completely satisfying. This is because the use of direct interval arithmetic to obtain the output of some continuous functions of fuzzy variable can lead to overestimation into the results of computation. The main drawback of using the direct interval arithmetic is that the formalism cannot represent the dependency among variables (Makino and Berz, 1999). Considering different occurrences of the same variable as independent can lead to repetition of some numerical computations. Then, there exist possible errors into computation. Eventually, the errors may produce approximations that are wider than the correct one (Bonarini and Bontempi, 1994).

Due to this, many researchers have proposed some other techniques such as the requisite constraint (Klir, 1997), the fuzzy weighted average (Dong and Wong, 1987; Yang *et al*., 1993), the vertex method (Dong and Shah, 1987), the transformation method (Hanss, 2002) and the spline approximation method (Chalco-Cano *et al*., 2009). However, the proposed methods increased computational complexity when applied to non-monotone functions as well. To overcome this deficiency, a new technique with better accuracy and low computational complexity should be investigated.

PRELIMINARIES

In this section, the basic idea of fuzzy sets will be introduced and some important concepts will be explained.

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Fuzzy sets: The notion of a fuzzy set is an extension of that of a classical or crisp set. Let X be a set of objects, called the universe, whose generic elements are denoted by x. Membership in a subset A of X can be viewed as a characteristic function, or membership function $A : X \to \{0, 1\}$ such that:

$$A(x) = \begin{cases} 1, \text{if } x \in A, \\ 0, \text{if } x \notin A. \end{cases} \quad (1)$$

The set {0, 1} is called the valuation set. If the valuation set is allowed to be the real unit interval [0, 1], then A is called a fuzzy subset of X or simply a fuzzy set in X. In this case, A(x) is interpreted as the degree of membership of the element x in the fuzzy set A.

Definition 1: Let A be a fuzzy set defined on $\Re$. The support of A is the crisp set of all points on $\Re$ such that the membership degree of A is non-zero, that is:

$$\text{supp}(A) = \{x \in \Re \; A(x) > 0\} \quad (2)$$

Definition 2: Let A be a fuzzy set defined on $\Re$. The core of A is the crisp set of all points on $\Re$ such that the membership degree of A is 1, that is:

$$\text{core}(A) = \{x \in \Re \; A(x) = 1\} \quad (3)$$

Definition 3: Let A be a fuzzy set defined on $\Re$. A is called a fuzzy interval if:

- A is normal: There exists $x_0 \in \Re$ such that $A(x_0) = 1$
- A is convex: for all $x, y \in \Re$ and, $0 \le \lambda \le 1$ it holds that:

$$A(\lambda x + (1-\lambda)y \ge \min(A(x), A(y))$$

- A is upper semi-continuous: for any $x_0 \in \Re$, it holds that:

$$A(x_0) \ge \lim_{x \to x_0^{\pm}} A(x)$$

- $[A]^0 = \overline{\{x \in \Re \mid A(x) \ge \alpha\}}$ is a compact subset of $\Re$.

The α-cut of a fuzzy interval A, with $0 < \alpha \le 1$ is the crisp set:

$$[A]^\alpha = \{x \in \Re \mid A(x) \ge \alpha\} \quad (4)$$

For a fuzzy interval A, its α-cuts are closed intervals in $\Re$; we denote them by:

$$[A]^\alpha = [a_1^\alpha, a_2^\alpha] \quad (5)$$

Definition 4: A fuzzy interval A is called a triangular fuzzy interval if its membership function has the following form:

$$A(x) = \begin{cases} 0, & \text{if } x < a, \\ \frac{x-a}{b-a}, & \text{if } a \le x \le b, \\ \frac{c-x}{c-b}, & \text{if } b \le x \le c, \\ 0, & \text{if } x > c, \end{cases} \quad (6)$$

and its α-cuts are simply:

$$[A]^\alpha = [a + \alpha(b-a), c - \alpha(c-b)], \alpha \in (0,1] \quad (7)$$

This definition asserts that the triangular fuzzy number A is defined by three numbers a < b < c where the core of the triangle is at x = b and its support is the interval (a, b). In this study the authors write A(a, b, c) for a triangular fuzzy interval. The set of all fuzzy intervals is denoted by $F(\Re)$.

Fuzzy interval arithmetic: In this subsection, the authors recall fuzzy interval arithmetic and present some of its operations. Arithmetic operations of fuzzy interval arithmetic are extensions of the operations of interval arithmetic introduced by Moore (1966).

Consider two fuzzy intervals A and B. The basic arithmetic operations of A and B are defined as follows:

- Addition:

$$[A+B]^\alpha = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha]$$

- Subtraction:

$$[A-B]^\alpha = [a_1^\alpha - b_2^\alpha, a_2^\alpha - b_1^\alpha]$$

- Multiplication:

$$[A \times B]^\alpha = [c_1^\alpha, c_2^\alpha]$$

where

$$c_1^\alpha = \min(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha)$$
$$c_2^\alpha = \max(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha)$$

- Division:

$$\left[\frac{A}{B}\right]^{\alpha}=[d_1^{\alpha},d_2^{\alpha}]$$

with $0\notin[B]^0$ where

$$d_1^{\alpha}=\min\left(\frac{a_1^{\alpha}}{b_2^{\alpha}},\frac{a_1^{\alpha}}{b_1^{\alpha}},\frac{a_2^{\alpha}}{b_2^{\alpha}},\frac{a_2^{\alpha}}{b_1^{\alpha}}\right)$$
$$d_2^{\alpha}=\max\left(\frac{a_1^{\alpha}}{b_2^{\alpha}},\frac{a_1^{\alpha}}{b_1^{\alpha}},\frac{a_2^{\alpha}}{b_2^{\alpha}},\frac{a_2^{\alpha}}{b_1^{\alpha}}\right)$$

The extension principle: It is useful to define functions on fuzzy sets. Any crisp function can be extended to take fuzzy set as its argument by applying Zadeh's extension principle (Zadeh, 1965). Let f be a function from X to Y. Given a fuzzy set A in X and want to find a fuzzy set B = f(A) in Y that is induced by f. If f is a monotone then B = f(A) can be determined as follow:

$$f(A)(y)=\begin{cases}A(f^{-1}(y)), & \text{if } x\in \text{range}(f)\\ 0, & \text{if } x\notin \text{range}(f)\end{cases} \tag{8}$$

It is clear that Eq. 8 can be easily calculated by determining the membership at the endpoints of the α-cuts of A. However, in general, the process of finding the fuzzy set B = f(A) is more complicated and cannot be implemented in a practical way. For example, if f is a non-monotone, then the problem arises when two or more distinct points in X are mapped to the same point in Y. If this is the case, the above Eq. may take two or more different values. This requires a new extension of Eq. 8 as enlisted below:

$$f(A)(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)} A(x), & \text{if } x\in \text{range}(f)\\ 0, & \text{if } x\notin \text{range}(f)\end{cases} \tag{9}$$

where,

$$f^{-1}(y)=\{x\in X\,|\,f(x)=y\}$$

The Eq. 9 is called Zadeh's extension principle. Roman-Flores *et al.* (2001) have shown that if f: X→Y is a continuous function, then f: F(X) →F(Y) is a well-defined function and:

$$[f(A)]^{\alpha}=f([A]^{\alpha}) \tag{10}$$

for all α∈[0, 1] and A∈F(X).

MATERIALS AND METHODS

The dependency problem: The standard operations of fuzzy interval arithmetic provide a manageable way to compute functions of fuzzy variable. However, the same fuzzy variable is considered independently in its operations. This characteristic can lead to overestimation in computation. To show this shortcoming, the authors give the following example.

Consider the triangular fuzzy interval U(-2, 0, 2). The corresponding α-cuts of U are $[U]^{\alpha}$ = [-2+2α, -2α+2] for α∈[0, 1]. The authors take the function $f:\Re\to\Re$ defined by $f(x) = x+x^2$ and want to find f(U). There are two common ways to obtain f(U). First, the authors apply the straightforward fuzzy interval arithmetic:

$$\begin{aligned}f([U]^{\alpha})&=[-2+2\alpha,-2\alpha+2]+[-2+2\alpha,-2\alpha+2]^2\\&=[-2+2\alpha,-2\alpha+2]+[d_1^{\alpha},d_2^{\alpha}]\end{aligned} \tag{11}$$

where,

$$d_1^{\alpha}=\min[(-2+2\alpha)^2,(-2+2\alpha)(-2\alpha+2)\\(-2\alpha+2)(-2+2\alpha),(-2\alpha+2)^2]$$

and:

$$d_2^{\alpha}=\max[(-2+2\alpha)^2,(-2+2\alpha)(-2\alpha+2)\\(-2\alpha+2)(-2+2\alpha),(-2\alpha+2)^2]$$

If α = 0, the solution is:

$$f([U]^0)=[-6,6] \tag{12}$$

For the second approach, the authors use Zadeh's extension principle by assuming that the variable U is independent:

$$f(U)=U+U^2 \tag{13}$$

For α = 0, the solution is therefore:

$$f([U]^0)=[-2,6] \tag{14}$$

Unfortunately, the both solutions are not the correct range of f(x) for all $x\in[U]^0$. To get the correct range of f(x), the authors define the right hand side of Eq. 13 as one single expression and from that the authors use Zadeh's extension principle and not part by part. By referring to the example discussed above and applying this idea to it then the new solution is:

$$f([U]^0) = [-0.25, 6] \tag{15}$$

which is the correct range of f(x) for all $x \in [U]^0$.

Discretisation of fuzzy intervals: There are two ways of discretising a fuzzy interval. Researchers can discretise either on the variable domain or on the membership value (Dong and Shah 1987). In this study, the authors concern the latter approach. One of the reasons is that the peak point, at which the membership degree is 1, is guaranteed to be included in the discretised counterparts. But this cannot be sure for any discretisation on the variable domain.

First, the authors discretise $\alpha \in [U]^0$ up to n finite numbers with length $\Delta\alpha = 1/(n-1)$. Hence, a set α is obtained as follow:

$$\alpha = \{\alpha_1, \alpha_2, ..., \alpha_i, ..., \alpha_n\} \tag{16}$$

where,

$$\alpha_1 = 0, \alpha_i = \alpha_{i-1} + \Delta\alpha$$

and $\alpha_n = 1$ for i = 2, 3 ... , n. Let A be a fuzzy interval with α-cuts denoted by:

$$[A]^\alpha = [a_1^\alpha, a_2^\alpha]$$

for $\alpha \in [0, 1]$. By discretising α up to n finite numbers, the authors get the following set of intervals:

$$I = \{[A]^{\alpha_1}, [A]^{\alpha_2}, ..., [A]^{\alpha_i}, ..., [A]^{\alpha_n}\} \tag{17}$$

For the different α-cuts of fuzzy interval A the following holds:

$$[A]^{\alpha_{i+1}} \subseteq [A]^{\alpha_i}, \forall \alpha_i, \alpha_{i+1} \in [0,1] \text{ with } \alpha_i \le \alpha_{i+1} \tag{18}$$

for i = 1, 2, ..., n-1. It is clear that the α-cuts of fuzzy interval A are connected. This means that the lower value of α will give wider intervals compared to the higher value of α. For each:

$$\alpha_i, [A]^{\alpha_i}$$

can be constructed as the union of sub-intervals according to the following Eq:

$$[A]^{\alpha_i} = [a_1^{\alpha_i}, a_1^{\alpha_{i+1}}] \cup [a_1^{\alpha_{i+1}}, a_2^{\alpha_{i+1}}] \cup [a_2^{\alpha_{i+1}}, a_2^{\alpha_i}] \tag{19}$$

for all $\alpha_i, \alpha_{i+1} \in [0,1]$ with $\alpha_i \le \alpha_{i+1}$ (Fig. 1).

Let f: ℜ→ℜ be a continuous function. Given a fuzzy interval A in ℜ and the authors want to find a fuzzy interval B = f(A) induced by f through Zadeh's extension principle. The authors compute:

$$[B]^{\alpha_i} = f([A]^{\alpha_i}) \tag{20}$$

at each level of α_i for i = 1, 2, ..., n according to the following Eq:

$$\hat{b}_{min}^{\alpha_i} = \min\left[\min_{x \in [a_1^{\alpha_i}, a_1^{\alpha_{i+1}}]} f(x), \min_{x \in [a_1^{\alpha_{i+1}}, a_2^{\alpha_{i+1}}]} f(x), \min_{x \in [a_2^{\alpha_{i+1}}, a_2^{\alpha_i}]} f(x)\right] \tag{21}$$

$$\hat{b}_{max}^{\alpha_i} = \max\left[\max_{x \in [a_1^{\alpha_i}, a_1^{\alpha_{i+1}}]} f(x), \max_{x \in [a_1^{\alpha_{i+1}}, a_2^{\alpha_{i+1}}]} f(x), \max_{x \in [a_2^{\alpha_{i+1}}, a_2^{\alpha_i}]} f(x)\right] \tag{22}$$

for i = 1, 2, ..., n-1. Here; $\hat{b}_{min}^{\alpha_i}$ and $\hat{b}_{max}^{\alpha_i}$ are the minimum and maximum values at points $x_{min}^{\alpha_i}$ and $x_{max}^{\alpha_i}$, respectively. The points $x_{min}^{\alpha_i}$ and $x_{max}^{\alpha_i}$ are defined as Eq. 23 and 24, respectively:

$$x_{min}^{\alpha_i} = \min[x \in [a_1^{\alpha_i}, a_2^{\alpha_i}] \mid f(x) = \hat{b}_{min}^{\alpha_i}] \tag{23}$$

and:

$$x_{max}^{\alpha_i} = \max[x \in [a_1^{\alpha_i}, a_2^{\alpha_i}] \mid f(x) = \hat{b}_{max}^{\alpha_i}] \tag{24}$$

The optimisation problems in Eq. 21 and 22 will be performed by using Brent's method (Brent, 2002). The goals are to find the minimum:

$$x_{min}^{\alpha_i} \in [a_1^{\alpha_i}, a_2^{\alpha_i}]$$

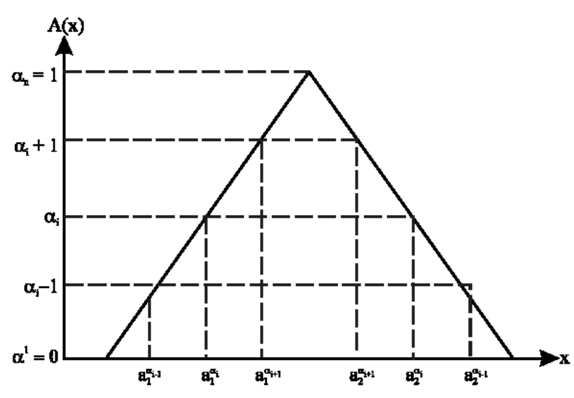

Fig. 1: α-discretisation of a fuzzy interval

and the maximum:

$$x_{max}^{\alpha_i} \in [a_1^{\alpha_i}, a_2^{\alpha_i}]$$

such that:

$$f(x) \geq f(x_{min}^{\alpha_i})$$

and:

$$f(x) \leq f(x_{max}^{\alpha_i})$$

for all:

$$x \in [a_1^{\alpha_i}, a_2^{\alpha_i}]$$

In order to increase the speed during computation, the authors propose a new strategy to find the minimum and the maximum values over the interval:

$$[a_1^{\alpha_i}, a_2^{\alpha_i}]$$

The authors start from the highest value of α and continue downward until the smallest value of α is reached. For instance, at α_i, there are three optimisation problems to be solved (Eq. 21 and 22). However, the second optimisation problem can be omitted since it has already been solved at α_{i+1}. For this, the authors only consider the first and the third optimisation problems. By taking the minimum (maximum) of all results of the optimisation problems, a new minimum value (a new maximum value) is obtained. The minimum (maximum) at α_i can be similar to or smaller (bigger) than the minimum (maximum) found at α_{i+1}, depending on the function under consideration. This process is repeated for all levels of α. Consequently, a set of intervals which finally turns out to be a fuzzy interval is obtained as enlisted in the following Eq:

$$O = \{[\hat{B}]^{\alpha_1}, [\hat{B}]^{\alpha_2}, \ldots, [\hat{B}]^{\alpha_i}, \ldots, [\hat{B}]^{\alpha_n}\} \quad (25)$$

Here:

$$[\hat{B}]^{\alpha_i} = [\hat{b}_{min}^{\alpha_i}, \hat{b}_{max}^{\alpha_i}]$$

where, $\hat{b}_{min}^{\alpha_i}$ and $\hat{b}_{max}^{\alpha_i}$ are defined as in Eq. 21 and 22, respectively.

Next, the authors introduce the following two errors:

Definition 5: Let $\Re \to \Re$ be a continuous function. Given a fuzzy interval A on $\Re$. The Horizontal-Error of B = f(A) is given by:

$$E_H^{\alpha_i} = |b_1^{\alpha_i} - \hat{b}_{min}^{\alpha_i}| + |b_2^{\alpha_i} - \hat{b}_{max}^{\alpha_i}|, i = 1, 2, \ldots, n, \quad (26)$$

where,

$$[B]^{\alpha_i} = [b_1^{\alpha_i}, b_2^{\alpha_i}] \text{ and } [\hat{B}]^{\alpha_i} = [\hat{b}_{min}^{\alpha_i}, \hat{b}_{max}^{\alpha_i}]$$

are the α-cuts of analytical solution and approximation solution, respectively.

Definition 6: Let $\Re \to \Re$ be a continuous function. Given a fuzzy interval A on $\Re$. The Vertical-Error of B= f(A) is given by:

$$E_V^{\alpha_i} = |f(A)(b_{min}^{\alpha_i}) - f(A)(\hat{b}_{min}^{\alpha_i})| + |f(A)(b_{max}^{\alpha_i}) - f(A)(\hat{b}_{max}^{\alpha_i})| \quad (27)$$

i = 1, 2 , ... , n, where, $f(A)(b_{min}^{\alpha_i})$ and $f(A)(b_{max}^{\alpha_i})$ are the membership values of analytical solution and $f(A)(\hat{b}_{min}^{\alpha_i})$ and $f(A)(\hat{b}_{max}^{\alpha_i})$ are the membership values of approximation solution.

Numerical example: In this section, the authors use the proposed method to illustrate the computation of Zadeh's extension principle for a continuous function.

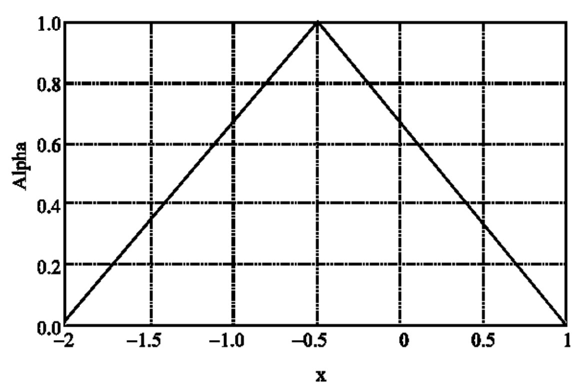

Fig. 2: Fuzzy interval U

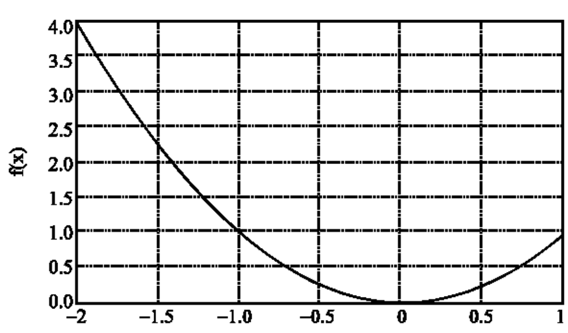

Fig. 3: Function Handle

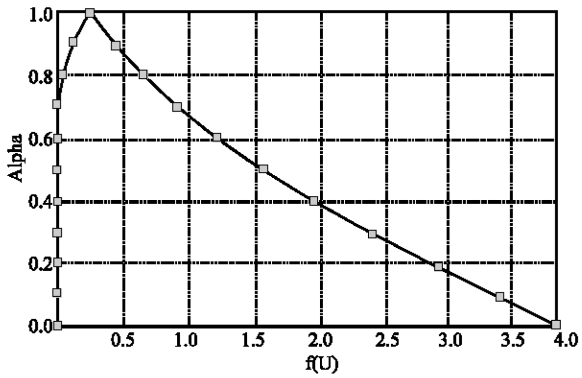

Fig. 4: Comparison between analytical solution (solid line) and its approximation (circle mark)

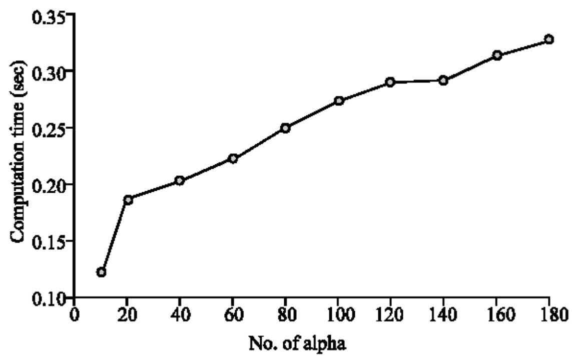

Fig. 5: Elapsed CPU time (sec)

The authors consider the following triangular fuzzy interval U defined by:

$$U(x)=\begin{cases} 0 & ,\text{if } x<-2, \\ \frac{2}{3}x+\frac{4}{3} & ,\text{if } -2\le x\le -\frac{1}{2}, \\ -\frac{2}{3}x+\frac{2}{3} & ,\text{if } -\frac{1}{2}\le x\le 1, \\ 0 & ,\text{if } x>1. \end{cases} \quad (28)$$

The α-cuts of U are given by:

$$[U]^{\alpha}=\left[\frac{3}{2}\alpha-2,-\frac{3}{2}\alpha+1\right] \quad (29)$$

The authors take the function $\Re\to\Re$ defined by:

$$f(x)=x^2 \quad (30)$$

and compute f(U) = $(U)^2$ through optimisation technique. Please note that the function f is unimodal defined over the interval [-2, 1]. It has an extreme point at x = 0. From Zadeh's extension principle, the analytical solution is given by:

Table 1: Approximation errors

α	Horizontal-error	Vertical-error
0.0	0.0000	0.0000
0.1	0.0000	0.0000
0.2	0.0000	0.0000
0.3	0.0000	0.0000
0.4	0.0000	0.0000
0.5	0.0000	0.0000
0.6	0.0000	0.0000
0.7	0.0000	0.0000
0.8	0.0000	0.0000
0.9	0.0000	0.0000
1.0	0.0000	0.0000

Table 2: Number of function evaluations

n	Our proposed method	The transformation method
11	61	66
21	121	231
41	241	861
61	361	1891
81	481	3321
101	601	5151

$$f(A)(y)=\begin{cases} 0,\text{ if } y<0, \\ \max\left(\frac{2}{3}\sqrt{y}+\frac{2}{3},-\frac{2}{3}\sqrt{y}+\frac{2}{3}\right),\text{ if } 0\le y\le \frac{1}{4}, \\ \max\left(-\frac{2}{3}\sqrt{y}+\frac{4}{3},-\frac{2}{3}\sqrt{y}+\frac{2}{3}\right),\text{ if } \frac{1}{4}\le y\le 1, \\ -\frac{2}{3}\sqrt{y}+\frac{4}{3},\text{ if } 1\le y\le 4, \\ 0,\ y>4 \end{cases} \quad (31)$$

The graphs of U, f(x) and f(U) are depicted in Fig. 2, 3 and 4, respectively. In Fig. 4, the authors compare the approximation solution with the analytical solution obtained via Zadeh's extension principle. The authors observe that the approximation solution equals to the analytical solution. The approximation errors (up to four decimal digits) are listed in Table 1.

In term of computational complexity, the authors observe that the total number of function evaluations required in this example is lower than the total number of function evaluations required in the transformation method (Table 2). In general, one can say that if the number of α is large, then the method proposed in this study offers a significantly better complexity.

Furthermore, the execution time (in elapsed CPU seconds) for the different numbers of α is mentioned in Fig. 5. It is consistent with the total number of function evaluations since the execution time increases as α increases.

CONCLUSION

The authors have presented a new method for computing a function that takes a fuzzy interval as its arguments. The main advantage of the proposed method is that it does not need a large computer memory and the speed up in computation is guaranteed. Moreover, it does not need information about derivative of a function. In the future, this proposed method will be incorporated into classical numerical methods for solving differential equations with fuzzy initial values.

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