Applied Mathematics and Computational Intelligence
Volume 11, No. 1, Dec 2022 [35 – 44]

The Approximate Solution for A Triangular Fully Fuzzy Matrix Equation $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$

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Received: 25 October 2021; Accepted: 9 November 2021; Available online (in press): 14 January 2022

ABSTRACT

A fully fuzzy matrix equation of $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$ has its own important in the application of control system engineering particularly in the situation of uncertainty. In this study, the equation is solved where the triangular fuzzy numbers be the variables of the equation. The algorithm that is used in the solution is consists of the conversion for the fully fuzzy matrix equation to a fully fuzzy linear system by utilizing the Kronecker and Vec-operator. Subsequently, the final solution is obtained by using the pseudoinverse method. A numerical example and the verification of the solution obtained are presented to demonstrate the contribution of this study.

Keywords: Control system engineering, Fully fuzzy matrix equation, Fully fuzzy linear system, Pseudoinverse method.

1 INTRODUCTION

There are variety of matrix equations that are generally used in many applications related with control system engineering [1–3]. According to [4], matrix equations play their function of equation solver for control system model. However, there is certain condition that could be uncertainty during the system process. Therefore, the investigation of the solution for the matrix equations with the parameters are in fuzzy numbers are become necessary [5].

The matrix equation with all the parameters are either in a triangular, trapezoidal or parametric form of fuzzy numbers is known as a fully fuzzy matrix equation (FFME). The study on the FFME of $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$ has been carried out since 2013 by [6], but the method proposed is quite tedious to be applied for the matrices with the size more than three. The number of studies keep increasing in a past few years for various forms of FFME, such as $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$ [7] and $\tilde{X}\tilde{A} = \tilde{C}$ [8]. Both studies have considering that the parameters and solutions of the FFME are in positive triangular fuzzy numbers (TFN). While, there are also studies have been carried out for solving FFME $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ by [9] and [10], where the parameters of the FFME are in parametric form and arbitrary form of trapezoidal fuzzy numbers, respectively. Recently, [11] proposed a modified fuzzy multiplication arithmetic operators in order to solve FFME of $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$, where the parameters are considering to be in a near-zero TFN.

Since, up to now, there is only one study conducted for solving $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$, hence, this study proposed a new algorithm for solving the FFME of

$$\tilde{A}\tilde{X}\tilde{B} = \tilde{C} \tag{1}$$

by considering that the parameters $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$, $\tilde{B} = (\tilde{b}_{ij})_{m\times n}$ and $\tilde{C} = (\tilde{c}_{ij})_{m\times n}$ are in a form of arbitrary TFN, while $\tilde{X} = (\tilde{x}_{ij})_{n\times m}$ is an approximate fuzzy solution. This study provides a simple and direct algorithm, which applicable for arbitrary TFN regardless to any size of matrices.

In solving the FFME, the Kronecker product and Vec-operator are used to transform the FFME to a fully fuzzy linear system (FFLS). In addition, an associated linear system (ALS) is used to formed the FFLS in a crisp form of linear system. Finally, the approximate fuzzy solution is obtained by using the pseudoinverse method.

This paper is organized as follows. In Section 2, some preliminaries that are used in this study is given. In Section 3, the algorithm that are developed for solving the FFME of Equation (1) is presented. A numerical example is illustrated in Section 4 and followed by the conclusion in Section 5.

2 PRELIMINARIES

In this section, some definitions and theories are reviewed which are used in this study.

2.1 Theory of Fuzzy Numbers

The definitions that describe on the theory of fuzzy numbers are given as follows which are based on [12–14].

Definition 1 *Let X be a nonempty set. A fuzzy set $\tilde{A}$ in X is characterized by its membership function*

$$\mu_{\tilde{A}} : X \to [0, 1] \tag{2}$$

and $\mu_{\tilde{A}}(x)$ represents the degree of membership of element x in fuzzy set $\tilde{A}$ for each $x \in X$.

Definition 2 *A fuzzy number $\tilde{M} = (m, \alpha, \beta)$ is said to be a triangular fuzzy number (TFN), if its membership function is given by:*

$$\mu_{\tilde{M}}(x) = \begin{cases} 1 - \frac{m-x}{\alpha}, & m - \alpha \le x \le m, \alpha > 0, \\ 1 - \frac{x-m}{\beta}, & m \le x \le m + \beta, \beta > 0, \\ 0, & \textit{otherwise}. \end{cases} \tag{3}$$

In this case, m is the mean value of $\tilde{M}$, whereas α and β are the right and left spreads, respectively.

Definition 3 *A fuzzy number $\tilde{M} = (m, \alpha, \beta)$ is called as an arbitrary TFN where it may be positive, negative or near zero which can be classified as follows:*

- *$\tilde{M}$ is a positive (negative) fuzzy number iff $m - \alpha \ge 0$ ($\beta + m \le 0$).*

- *$\tilde{M}$ is a near zero fuzzy number iff $m - \alpha \leq 0 \leq \beta + m$.*

Usually, the multiplication operation for the TFN is executed based on the fuzzy arithmetic operator introduced by [13]. However, since the Dubois's multiplication operator is limited for the arbitrary fuzzy numbers, then in this study, the multiplication operation is executed based on the following operators.

Definition 4 *[15] The product of two triangular fuzzy numbers $\tilde{M} = (m, \alpha, \beta)$ and $\tilde{N} = (n, \gamma, \delta)$, is defined as*

$$\tilde{M} \otimes \tilde{N} = \begin{cases} (mn, f_1, f_2) & (m, \alpha, \beta) \geq 0 \\ (mn, f_3, f_4) & (m, \alpha, \beta) \leq 0 \\ (mn, f_5, f_6) & otherwise. \end{cases} \tag{4}$$

where
$f_1 = mn - Min\big((m-\alpha)(n-\gamma), (m+\beta)(n-\gamma)\big)$,
$f_2 = Max\big((m-\alpha)(n+\delta), (m+\beta)(n+\delta)\big) - mn$,
$f_3 = mn - Min\big((m-\alpha)(n+\delta), (m+\beta)(n+\delta)\big)$,
$f_4 = Max\big((m-\alpha)(n-\gamma), (m+\beta)(n-\gamma)\big) - mn$,
$f_5 = mn - Min\big((m-\alpha)(n+\delta), (m+\beta)(n-\gamma)\big)$,
$f_6 = Max\big((m-\alpha)(n-\gamma), (m+\beta)(n+\delta)\big) - mn$.

2.2 Classical Linear Systems

A classical linear system of equations which generally denoted as $AX = B$, is a set or collection of equations which consists of similar set variables [16]. Normally, the classical linear system can be solved using direct inverse method, such that

$$X = A^{-1}B \tag{5}$$

where A is an invertible coefficient matrix and X is a unique solution. On the other hand, if A is non-invertible, then the system can be solved by pseudoinverse method.

Definition 5 *[17] If $A \in \mathbb{M}^{m\times n}$, then there exists a unique $A^\dagger \in \mathbb{M}^{m\times n}$, such that*

$$A^\dagger = (A^T A)^{-1} A^T. \tag{6}$$

where $A^\dagger$ is the pseudoinverse of matrix A.

Thus, the solution for $AX = B$ can be obtained by

$$X = A^\dagger B, \tag{7}$$

where X is a best approximate solution of the linear system.

2.3 Fundamental Concepts of Fuzzy Kronecker Product and Fuzzy Vec-operator

The function of the Kronecker product and Vec-operator are also important in this study in converting the FFME to a simpler form of equations.The definitions and theorems of the Kronecker product and Vec-operator in the fuzzy numbers environment are provided as follows:

Definition 6 *[18] Let* $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$ *and* $\tilde{B} = (\tilde{b}_{ij})_{p\times q}$ *be fuzzy matrices. Fuzzy Kronecker product is represented as* $\tilde{A} \otimes_k \tilde{B}$*, where*

$$\tilde{A} \otimes_k \tilde{B} = \begin{pmatrix} \tilde{a}_{11}\tilde{B} & \tilde{a}_{12}\tilde{B} & \dots & \tilde{a}_{1n}\tilde{B} \\ \tilde{a}_{21}\tilde{B} & \tilde{a}_{22}\tilde{B} & \dots & \tilde{a}_{2n}\tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1}\tilde{B} & \tilde{a}_{m2}\tilde{B} & \dots & \tilde{a}_{mn}\tilde{B} \end{pmatrix} = [\tilde{a}_{ij}\tilde{B}]_{(mp)\times(nq)} \tag{8}$$

Definition 7 *[18] Vec-operator of a fuzzy matrix is a linear transformation that converts the fuzzy matrix of* $\tilde{C} = (\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_n)$ *into a column vector as*

$$Vec(\tilde{C}) = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_n \end{pmatrix}. \tag{9}$$

Theorem 1 *[18] If* $\tilde{A} = (\tilde{a}_{ij})_{m\times m}$ *is a fuzzy matrix, and* $\tilde{U} = (\tilde{u}_{ij})_{p\times p}$ *is a unitary fuzzy matrix defined as*

$$\tilde{U} = \begin{pmatrix} (1,0,0) & (0,0,0) & \dots & (0,0,0) \\ (0,0,0) & (1,0,0) & \dots & (0,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0) & (0,0,0) & \dots & (1,0,0) \end{pmatrix}, \tag{10}$$

then

i. $\tilde{A}\tilde{U} = \tilde{U}\tilde{A} = \tilde{A}$

ii. $\tilde{U}^T = \tilde{U}$.

Definition 8 *[18] Let* $A = (a_{ij})_{m\times m}$*,* $B = (b_{ij})_{n\times n}$ *and* $X = (x_{ij})_{m\times n}$*, then*

i. $Vec[\tilde{A}\tilde{X}] = [\tilde{U}_n \otimes_k \tilde{A}]Vec(\tilde{X})$

ii. $Vec[\tilde{X}\tilde{B}] = [\tilde{B}^T \otimes_k \tilde{U}_m]Vec(\tilde{X})$

3 ALGORITHM FOR SOLVING $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$

In this section, the developed algorithm is presented. First, the FFME of $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$ is converted to a form of FFLS, denoted as $\tilde{S}\tilde{X} = \tilde{C}$, where $\tilde{S} = [\tilde{B}^T \otimes_k \tilde{A}]$, $\tilde{X} = Vec(\tilde{X})$ and $\tilde{C} = Vec(\tilde{C})$. The conversion is based on the following Lemma 1 and Theorem 2.

Lemma 1 *Let* $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$ *and* $\tilde{B} = (\tilde{b}_{ij})_{m\times n}$*, then*

$$\tilde{A}_{m\times n} \otimes_k \tilde{B}_{m\times n} = [(\tilde{A}_{m\times n} \otimes_k \tilde{U}_{n\times n})(\tilde{U}_{m\times m} \otimes_k \tilde{B}_{m\times n})] \tag{11}$$

Proof According to Definition 6 and the standard matrix multiplication,

$$\tilde{A}_{m\times n} \otimes_k \tilde{B}_{m\times n} = \begin{pmatrix} \tilde{a}_{11}\tilde{B} & \tilde{a}_{12}\tilde{B} & \dots & \tilde{a}_{1n}\tilde{B} \\ \tilde{a}_{21}\tilde{B} & \tilde{a}_{22}\tilde{B} & \dots & \tilde{a}_{2n}\tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1}\tilde{B} & \tilde{a}_{m2}\tilde{B} & \dots & \tilde{a}_{mn}\tilde{B} \end{pmatrix}.$$

On the other hand, since

$$(\tilde{A}_{m\times n} \otimes_k \tilde{U}_{n\times n}) = \begin{pmatrix} \tilde{a}_{11}\tilde{U} & \tilde{a}_{12}\tilde{U} & \dots & \tilde{a}_{1n}\tilde{U} \\ \tilde{a}_{21}\tilde{U} & \tilde{a}_{22}\tilde{U} & \dots & \tilde{a}_{2n}\tilde{U} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1}\tilde{U} & \tilde{a}_{m2}\tilde{U} & \dots & \tilde{a}_{mn}\tilde{U} \end{pmatrix}$$

and

$$(\tilde{U}_{m\times m} \otimes_k \tilde{B}_{m\times n}) = \begin{pmatrix} \tilde{B} & 0 & \dots & 0 \\ 0 & \tilde{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{B} \end{pmatrix}.$$

Then,

$$\begin{aligned} (\tilde{A}_{m\times n} \otimes_k \tilde{U}_{n\times n})(\tilde{U}_{m\times m} \otimes_k \tilde{B}_{m\times n}) &= \begin{pmatrix} \tilde{a}_{11}\tilde{U} & \tilde{a}_{12}\tilde{U} & \dots & \tilde{a}_{1n}\tilde{U} \\ \tilde{a}_{21}\tilde{U} & \tilde{a}_{22}\tilde{U} & \dots & \tilde{a}_{2n}\tilde{U} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1}\tilde{U} & \tilde{a}_{m2}\tilde{U} & \dots & \tilde{a}_{mn}\tilde{U} \end{pmatrix}\begin{pmatrix} \tilde{B} & 0 & \dots & 0 \\ 0 & \tilde{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{B} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{a}_{11}\tilde{B} & \tilde{a}_{12}\tilde{B} & \dots & \tilde{a}_{1n}\tilde{B} \\ \tilde{a}_{21}\tilde{B} & \tilde{a}_{22}\tilde{B} & \dots & \tilde{a}_{2n}\tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1}\tilde{B} & \tilde{a}_{m2}\tilde{B} & \dots & \tilde{a}_{mn}\tilde{B} \end{pmatrix}. \end{aligned}$$

It is shown that the expressions on the left-hand side and the right-hand side are equal. Hence, the lemma is proof. □

Theorem 2 *Let $\tilde{A}$ and $\tilde{B}$ be any size of fuzzy matrices, then the FFME of $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$ is equivalent to FFLS of*

$\tilde{S}\tilde{X} = \tilde{C}$,

where $\tilde{S} = [\tilde{B}^T \otimes_k \tilde{A}]$, $\tilde{X} = Vec(\tilde{X})$ and $\tilde{C} = Vec(\tilde{C})$.

Proof By considering the Vec-operator to $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$,

$$Vec(\tilde{A}\tilde{X}\tilde{B}) = Vec(\tilde{C}), \tag{12}$$

then

$$\begin{aligned} Vec(\tilde{A}\tilde{X}\tilde{B}) &= Vec(\tilde{A}\tilde{X}\tilde{U}^T\tilde{B}) && \text{by Theorem 1(i) and (ii)} \\ &= Vec([\tilde{A}\tilde{X}][\tilde{B}\tilde{U}^T]) && \text{by Theorem 1(i)} \\ &= Vec([\tilde{A}\tilde{X}][\tilde{B}^T\tilde{U}]^T) && \text{by properties of matrix transpose} \\ &= (\tilde{B}^T\tilde{U} \otimes_k \tilde{U})Vec(\tilde{A}\tilde{X}) && \text{by Definition 8(ii)} \\ &= (\tilde{B}^T \otimes_k \tilde{U})Vec(\tilde{A}\tilde{X}) && \text{by Theorem 1(i)} \\ &= (\tilde{B}^T \otimes_k \tilde{U})[(\tilde{U} \otimes_k \tilde{A})Vec(\tilde{X})] && \text{by Definition 8(i)} \\ &= [(\tilde{B}^T \otimes_k \tilde{U})(\tilde{U} \otimes_k \tilde{A})]Vec(\tilde{X}) && \text{by associative matrix} \\ &= [\tilde{B}^T \otimes_k \tilde{A}]Vec(\tilde{X}) && \text{by Lemma 1} \end{aligned}$$

Hence

$$\left[\tilde{B}^T \otimes_k \tilde{A}\right] Vec(\tilde{X}) = Vec(\tilde{C})$$

which is equivalent to

$$\tilde{S}\tilde{X} = \tilde{C}, \tag{13}$$

where $\tilde{S} = [\tilde{B}^T \otimes_k \tilde{A}]$, $\tilde{X} = Vec(\tilde{X})$ and $\tilde{C} = Vec(\tilde{C})$. □

Next, the FFLS of Equation (13) is solved by using an associated linear system (ALS), which has been built based on fuzzy arithmetic operator. The definition of the ALS is given as follows.

Definition 9 *Let the FFLS of* $\tilde{S}\tilde{X} = \tilde{C}$ *such that* $\tilde{S} = (m^{\tilde{S}}, \alpha^{\tilde{S}}, \beta^{\tilde{S}})$, $\tilde{C} = (m^{\tilde{C}}, \alpha^{\tilde{C}}, \beta^{\tilde{C}})$ *and* $\tilde{X} = (m^{\tilde{X}}, \alpha^{\tilde{X}}, \beta^{\tilde{X}})$ *are arbitrary fuzzy matrices. Then, the ALS* $SX = C$ *is formed as:*

$$\left(\begin{array}{c|c|c} m^{\tilde{S}} & 0 & 0 \\ \hline -\beta^{\tilde{S}} & (m^{\tilde{S}}+\beta^{\tilde{S}})^{+} & -(m^{\tilde{S}}+\beta^{\tilde{S}})^{-} \\ \hline -\alpha^{\tilde{S}} & -(m^{\tilde{S}}-\alpha^{\tilde{S}})^{-} & (m^{\tilde{S}}-\alpha^{\tilde{S}})^{+} \end{array}\right)\left(\begin{array}{c} m^{\tilde{X}} \\ \hline \alpha^{\tilde{X}} \\ \hline \beta^{\tilde{X}} \end{array}\right) = \left(\begin{array}{c} m^{\tilde{C}} \\ \hline \alpha^{\tilde{C}} \\ \hline \beta^{\tilde{C}} \end{array}\right) \tag{14}$$

where $(m^{\tilde{S}} - \alpha^{\tilde{S}})^{+}$ *and* $(m^{\tilde{S}} + \beta^{\tilde{S}})^{+}$ *contain the positive elements of* $(m^{\tilde{S}} - \alpha^{\tilde{S}})$ *and* $(m^{\tilde{S}} + \beta^{\tilde{S}})$ *respectively, while the negative elements are replaced by zero value. Similar to* $(m^{\tilde{S}} - \alpha^{\tilde{S}})^{-}$ *and* $(m^{\tilde{S}} + \beta^{\tilde{S}})^{-}$ *which contain the negative elements of* $(m^{\tilde{S}} - \alpha^{\tilde{S}})$ *and* $(m^{\tilde{S}} + \beta^{\tilde{S}})$ *respectively, while the positive elements are replaced by zero value.*

In obtaining the final solution, the coefficient S of the ALS in Equation (14) can be inverse directly, so that the fuzzy solution $\tilde{X}$ can be obtained. However, in this study, the coefficient S is considering as non-invertible matrix, thus the pseudoinverse method as in Definition 5 is used in obtaining the fuzzy approximation solution, such that

$$X = S^{\dagger}C. \tag{15}$$

where $S^{\dagger}$ is the pseudoinverse of matrix S.

4 NUMERICAL EXAMPLE

Example 1 *Consider the following FFME of* $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$

$$\begin{pmatrix} (-3,1,7) \\ (-2,4,10) \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \end{pmatrix} \otimes \begin{pmatrix} (9,2,12) & (2,1,3) \\ (6,3,13) & (12,2,7) \\ (11,4,8) & (9,4,9) \end{pmatrix} = \begin{pmatrix} (420,2376,1536) & (327,1787,1133) \\ (280,4192,2654) & (218,3138,1972) \end{pmatrix}$$

where the coefficients $\tilde{A}$ *and* $\tilde{B}$ *are near-zero and positive TFN respectively, while* $\tilde{X}$ *is a fuzzy solution.*

Solution:

The solution begins by converting the FFME to FFLS.

$$\tilde{B}^T \otimes_k \tilde{A} = \begin{pmatrix} (9,2,12) & (6,3,13) & (11,4,8) \\ (2,1,3) & (12,2,7) & (9,4,9) \end{pmatrix} \otimes_k \begin{pmatrix} (-3,1,7) \\ (-2,4,10) \end{pmatrix}$$

$$= \begin{pmatrix} (-27,57,111) & (-18,58,94) & (-33,43,109) \\ (-18,108,186) & (-12,102,164) & (-22,92,174) \\ (-6,14,26) & (-36,40,112) & (-27,45,99) \\ (-4,26,44) & (-24,90,176) & (-18,90,162) \end{pmatrix}$$

From that, the FFLS of $\tilde{S}\tilde{X} = \tilde{C}$ is

$$\begin{pmatrix} (-27,57,111) & (-18,58,94) & (-33,43,109) \\ (-18,108,186) & (-12,102,164) & (-22,92,174) \\ (-6,14,26) & (-36,40,112) & (-27,45,99) \\ (-4,26,44) & (-24,90,176) & (-18,90,162) \end{pmatrix} \begin{pmatrix} (m_{11}^{\tilde{X}}, \alpha_{11}^{\tilde{X}}, \beta_{11}^{\tilde{X}}) \\ (m_{12}^{\tilde{X}}, \alpha_{12}^{\tilde{X}}, \beta_{12}^{\tilde{X}}) \\ (m_{13}^{\tilde{X}}, \alpha_{13}^{\tilde{X}}, \beta_{13}^{\tilde{X}}) \end{pmatrix} = \begin{pmatrix} (420,2376,1536) \\ (280,4192,2654) \\ (327,1787,1133) \\ (218,3138,1972) \end{pmatrix}.$$

In order to transform the FFLS to the ALS, the coefficient of the FFLS are converted to the crisp form of matrices, as follows:

$$m^{\tilde{S}} = \begin{pmatrix} -27 & -18 & -33 \\ -18 & -12 & -22 \\ -6 & -36 & -27 \\ -4 & -24 & -18 \end{pmatrix}, \quad \alpha^{\tilde{S}} = \begin{pmatrix} 57 & 58 & 43 \\ 108 & 102 & 92 \\ 14 & 40 & 45 \\ 26 & 90 & 90 \end{pmatrix}, \quad \beta^{\tilde{S}} = \begin{pmatrix} 111 & 94 & 109 \\ 186 & 164 & 174 \\ 26 & 112 & 99 \\ 44 & 176 & 162 \end{pmatrix}$$

and the left-hand side $\tilde{C}$ is extracted to be as follows:

$$m^{\tilde{C}} = \begin{pmatrix} 420 \\ 280 \\ 327 \\ 218 \end{pmatrix}, \alpha^{\tilde{C}} = \begin{pmatrix} 2376 \\ 4192 \\ 1787 \\ 3138 \end{pmatrix}, \beta^{\tilde{C}} = \begin{pmatrix} 1536 \\ 2654 \\ 1133 \\ 1972 \end{pmatrix}.$$

Additionally,

$$(m^{\tilde{S}} - \alpha^{\tilde{S}}) = \begin{pmatrix} -84 & -76 & -76 \\ -126 & -114 & -114 \\ -20 & -76 & -72 \\ -30 & -114 & -108 \end{pmatrix},$$

$$(m^{\tilde{S}} - \alpha^{S})^{+} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; (m^{\tilde{S}} - \alpha^{\tilde{S}})^{-} = \begin{pmatrix} -84 & -76 & -76 \\ -126 & -114 & -114 \\ -20 & -76 & -72 \\ -30 & -114 & -108 \end{pmatrix}.$$

On the other hand,

$$(m^{\tilde{S}} + \beta^{\tilde{S}}) = \begin{pmatrix} 84 & 76 & 76 \\ 168 & 152 & 152 \\ 20 & 76 & 72 \\ 40 & 152 & 144 \end{pmatrix},$$

$$(m^{\tilde{S}} + \beta^{\tilde{S}})^{+} = \begin{pmatrix} 84 & 76 & 76 \\ 168 & 152 & 152 \\ 20 & 76 & 72 \\ 40 & 152 & 1446 \end{pmatrix}; (m^{\tilde{S}} + \beta^{\tilde{S}})^{-} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, the ALS which is in the form of $SX = C$ as shown in Definition 14 is performed as

$$\left(\begin{array}{ccc|ccc|ccc} -27 & -18 & -33 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & -12 & -22 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -36 & -27 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -24 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -111 & -94 & -109 & 84 & 76 & 76 & 0 & 0 & 0 \\ -186 & -164 & -174 & 168 & 152 & 152 & 0 & 0 & 0 \\ -26 & -112 & -99 & 20 & 76 & 72 & 0 & 0 & 0 \\ -44 & -176 & -162 & 40 & 152 & 144 & 0 & 0 & 0 \\ \hline -57 & -58 & -43 & 84 & 76 & 76 & 0 & 0 & 0 \\ -108 & -102 & -92 & 126 & 114 & 114 & 0 & 0 & 0 \\ -14 & -40 & -45 & 20 & 76 & 72 & 0 & 0 & 0 \\ -26 & -90 & -90 & 30 & 114 & 108 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} m_{1,1}^{\tilde{X}} \\ m_{1,2}^{\tilde{X}} \\ m_{1,3}^{\tilde{X}} \\ \hline \alpha_{1,1}^{\tilde{X}} \\ \alpha_{1,2}^{\tilde{X}} \\ \alpha_{1,3}^{\tilde{X}} \\ \hline \beta_{1,1}^{\tilde{X}} \\ \beta_{1,2}^{\tilde{X}} \\ \beta_{1,3}^{\tilde{X}} \end{array}\right) = \left(\begin{array}{c} 420 \\ 280 \\ 327 \\ 218 \\ \hline 2376 \\ 4192 \\ 1787 \\ 3138 \\ \hline 1536 \\ 2654 \\ 1133 \\ 1972 \end{array}\right).$$

In this case, since the coefficient of the ALS is non-invertible matrix, thus, the pseudoinverse method as stated in (15) is applied, hence

$$X=\begin{pmatrix}-5.41699\\-3.31542\\-6.48678\\1.68041\\4.13539\\3.95475\\0\\0\\0\end{pmatrix} \text{ or } X=\begin{pmatrix}\begin{pmatrix}m^{\tilde{X}}_{1,1}\\m^{\tilde{X}}_{1,2}\\m^{\tilde{X}}_{1,3}\end{pmatrix}\\\begin{pmatrix}\alpha^{\tilde{X}}_{1,1}\\\alpha^{\tilde{X}}_{1,2}\\\alpha^{\tilde{X}}_{1,3}\end{pmatrix}\\\begin{pmatrix}\beta^{\tilde{X}}_{1,1}\\\beta^{\tilde{X}}_{1,2}\\\beta^{\tilde{X}}_{1,3}\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}-5.41699\\-3.31542\\-6.48678\end{pmatrix}\\\begin{pmatrix}1.68041\\4.13539\\3.95475\end{pmatrix}\\\begin{pmatrix}0\\0\\0\end{pmatrix}\end{pmatrix}. \tag{16}$$

Hence, the solution obtained is an approximate negative fuzzy solution $\tilde{X}$, which is

$$\begin{aligned}\tilde{X}&=\left((m^{\tilde{X}}_{1,1},\alpha^{\tilde{X}}_{1,1},\beta^{\tilde{X}}_{1,1})\quad(m^{\tilde{X}}_{1,2},\alpha^{\tilde{X}}_{1,2},\beta^{\tilde{X}}_{1,2})\quad(m^{\tilde{X}}_{1,3},\alpha^{\tilde{X}}_{1,3},\beta^{\tilde{X}}_{1,3})\right)\\&=((-5.41699,1.68041,0)\quad(-3.31542,4.13539,0)\quad(-6.48678,3.95475,0)).\end{aligned}$$

Furthermore, a verification of the obtained solution is implemented by substituting the solution $\tilde{X}$ to the to the left-hand side of FFME as stated in Example 1.

$$\begin{aligned}\tilde{A}\tilde{X}&=\begin{pmatrix}(-3,1,7)\\(-2,4,10)\end{pmatrix}\otimes((-5.41699,1.68041,0)(-3.31542,4.13539,0)(-6.48678,3.95475,0))\\&=\begin{pmatrix}(16.251,44.6406,12.1386)\,(9.94626,39.7495,19.857)\,(19.4603,61.2265,22.3058)\\(10.834,67.6132,31.7504)\,(6.63084,66.2373,38.074)\,(12.9736,96.5058,49.6756)\end{pmatrix}.\end{aligned}$$

Subsequently,

$$\begin{aligned}\tilde{A}\tilde{X}\tilde{B}&=\begin{pmatrix}(16.251,44.6406,12.1386)\,(9.94626,39.7495,19.857)\,(19.4603,61.2265,22.3058)\\(10.834,67.6132,31.7504)\,(6.63084,66.2373,38.074)\,(12.9736,96.5058,49.6756)\end{pmatrix}\\&\otimes\begin{pmatrix}(9,2,12)&(2,1,3)\\(6,3,13)&(12,2,7)\\(11,4,8)&(9,4,9)\end{pmatrix}\\&=\begin{pmatrix}(420,2376,1536)\,(327,1787,1133)\\(280,4192,2654)\,(218,3138,1972)\end{pmatrix}\\&=\tilde{C}\end{aligned}$$

which is equal to $\tilde{C}$, the matrix at the right hand side of the FFME. Therefore, the solution is verified.

5 CONCLUSION

This study presents an algorithm for solving the FFME of $\tilde{A}\tilde{X}\tilde{B}=\tilde{C}$, where the parameters are arbitrary TFN. The algorithm utilizes the Kronecker product and Vec-operator in transforming the FFME to FFLS and forming the crisp form of linear system based on ALS. By considering that the FFME involves with a non-invertible coefficient matrix, then the pseudoinverse method is applied. Therefore, an approximate solution is obtained. For the future research, the other type of linear and non-linear matrix equations will be considered, such as $AX+XA^T=C$, $AXA^T-X=C$ and $AXB+CXD=E$, since the equations are also crucial in the real control system applications.

ACKNOWLEDGEMENT

The authors wish to thank the organizer of ICMS2021 and AMCI for accepting our paper and also to the reviewer for the constructive comments to improve this paper.

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